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Technical Report No. 32-95

The Doppler Shift For Accelerating Transmitters, Reflectors, and Receivers

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THE DOPPLER SHIFT FOR ACCELERATING TRANSMITTERS, REFLECTORS, AND RECEIVERS

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ABSTRACT

Several experiments utilizing two-way doppler measurements of signals reflected from the surface of Venus are currently under way. Since neither the Earth nor Venus is an inertial frame of reference, the usual formulas for the doppler shift do not apply. In this report, the equations for the calculation of the shift are developed when the transmitter, reflector, and receiver are accelerating. The results are easily generalized to the case where the transmitter and receiver are not coincident on the accelerating body.

I. INTRODUCTION

It has recently been proposed that more accurate values for the astronomical unit and the speed of light may be obtained by means of an experiment utilizing two-way doppler measurements reflected from the surface of Venus. Since neither the Earth nor Venus is an inertial frame of reference, the usual formulae for the doppler shift with (special) relativistic corrections do not apply. It is the purpose of this paper to obtain more accurate formulae for the phenomenon and to set down a general mathematical framework by means of which one may derive correct results without recourse to those methods applicable only to special relativity.

II. ONE-WAY DOPPLER

Let $0-x-y-z$ and $\bar{0}-\bar{x}-\bar{y}-\bar{z}$ be inertial frames of reference, and let $\mathbf{v}$ be the velocity of $\bar{0}$ relative to 0. The special theory of relativity states that

$$\begin{aligned} ds^2 &= c^2\,dt^2 - dx^2 - dy^2 - dz^2 \\ &= c^2\,d\bar{t}^{\,2} - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 \end{aligned} \tag{1}$$

is the fundamental invariant as regards space-time measurements. A clock fixed in the $\bar{0}-\bar{x}-\bar{y}-\bar{z}$ frame ($\bar{x}$, $\bar{y}$, and $\bar{z}$ being constant) measures the invariant

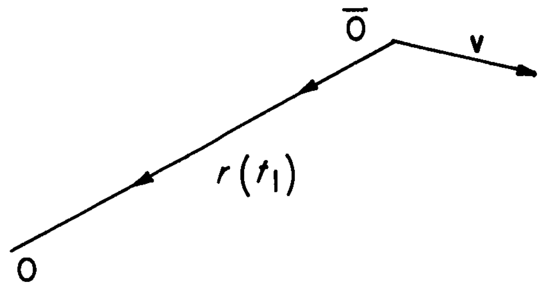
$$\begin{aligned} ds^2 = c^2\,d\bar{t}^{\,2} &= c^2\left\{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right\} dt^2 \\ &= c^2\left(1 - \frac{v^2}{c^2}\right) dt^2 \end{aligned} \tag{2}$$

so that

$$dt = \frac{d\bar{t}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} > d\bar{t}\,, \quad v^2 > 0 \tag{3}$$

Equation (3) states that observers fixed in the $0-x-y-z$ frame of reference will note that a moving clock (at rest in the $\bar{0}-\bar{x}-\bar{y}-\bar{z}$ frame of reference, but moving with velocity $\mathbf{v}$ relative to the $0-x-y-z$ frame of reference) will run at a slower rate ($dt > d\bar{t}$). Moreover, Eq. (3) holds for any clock instantaneously at rest in the $\bar{0}-\bar{x}-\bar{y}-\bar{z}$ frame of reference.

Now, let light signals emanate from $\bar{0}$ which are received at 0 (see Sketch 1):



Sketch 1

A signal emitted at $\bar{0}$ at time t_1 (t_1 is the time of emission as seen by an $0 - x - y - z$ observer) reaches 0 at time t_2 given by

$$t_2 = t_1 + \frac{r(t_1)}{c} \tag{4}$$

Thus

$$dt_2 = dt_1 \left[1 + \frac{1}{c} \frac{d\,r(t_1)}{dt} \right] \tag{5}$$

where dt_1 is the time elapsed between two successive peaks of a continuously emitting light source (monochromatic) and dt_2 is the difference in the time of reception at 0 of these peaks. From Eq. (3), $dt_1 = d\bar{t} / \sqrt{1 - v^2/c^2}$, with $d\bar{t}$ the time between successive peaks as noted by $\bar{0}$ moving with the source. Hence

$$\frac{dt_2}{d\bar{t}} = \frac{1 + \dfrac{1}{c} \dfrac{d\,r(t_1)}{dt}}{\left(1 - \dfrac{v^2}{c^2}\right)^{1/2}} \tag{6}$$

Since frequencies are inversely proportional to the times between successive peaks, one obtains

$$\nu_T = \nu_R \frac{1 + \frac{\dot{r}}{c}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \tag{7}$$

with $\dot{r}$ the radial speed of the source relative to 0. For $\Delta\nu = \nu_R - \nu_T$, it follows that

$$\Delta\nu = \nu_R \left[1 - \frac{1 + \frac{\dot{r}}{c}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}\right] \tag{8}$$

One can obtain Eq. (8) by simpler means by use of special relativity concepts. However, the technique used above applies equally well to noninertial frames of references treated in the next section.

It should be pointed out that Eq. (5) is only approximate. Another term in the Taylor's series expansion yields

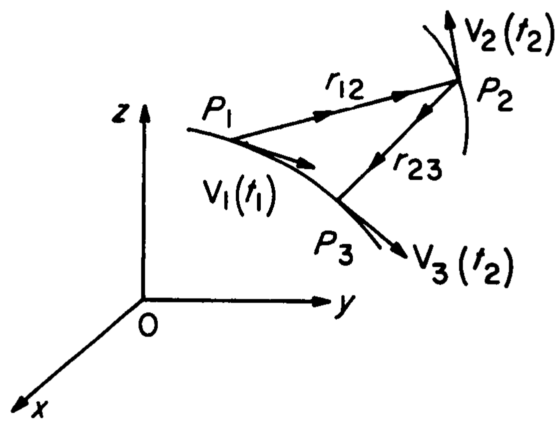
$$\Delta t_2 = \Delta t_1 \left[1 + \frac{1}{c}\dot{r} + \frac{1}{2c}\frac{d^2r}{dt^2}\Delta t_1\right] \tag{9}$$

or

$$\nu_T \approx \nu_R \frac{1 + \frac{\dot{r}}{c} + \frac{1}{2c\,\nu_T}\frac{d^2r}{dt^2}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \tag{10}$$

III. TWO-WAY DOPPLER

Let $0-x-y-z$ be an inertial frame of reference, and let P_1 be a moving source whose signals are received at a moving target P_2, which in turn reflects the signal back to $P_3 = P_1$ (see Sketch 2):



Sketch 2

Subsequently, all times will refer to the inertial frame of reference $0-x-y-z$. A signal emitted at P_1 at time t_1 will reach P_2 at time t_2 given by

$$t_2 = t_1 + \frac{1}{c} r_{12}(t_1, t_2) = t_1 + \frac{1}{c} \left\{[x_2(t_2) - x_1(t_1)]^2 + [y_2(t_2) - y(t_1)]^2 + [z_2(t_2) - z_1(t_1)]^2\right\}^{½} \tag{11}$$

Thus

$$t_2 + \Delta t_2 = t_1 + \Delta t_1 + \frac{1}{c} r_{12}(t_1 + \Delta t_1, t_2 + \Delta t_2)$$

$$\Delta t_2 = \Delta t_1 + \frac{1}{c} [r_{12}(t_1 + \Delta t_1, t_2 + \Delta t_2) - r_{12}(t_1, t_2)] \tag{12}$$

$$= \Delta t_1 + \frac{1}{c} \left\{ \frac{\partial r_{12}}{\partial t_1} \Delta t_1 + \frac{\partial r_{12}}{\partial t_2} \Delta t_2 + \frac{1}{2} \left[\frac{\partial^2 r_{12}}{\partial t_1^2} (\Delta t_1)^2 + 2 \frac{\partial^2 r_{12}}{\partial t_1 \partial t_2} \Delta t_1 \Delta t_2 + \frac{\partial^2 r_{12}}{\partial t_2^2} (\Delta t_2)^2 \right] \right\}$$

neglecting higher order terms, so that

$$\frac{\Delta t_2}{\Delta t_1} = \frac{1 + \dfrac{1}{c}\dfrac{\partial r_{12}}{\partial t_1} + \dfrac{1}{2c}\left(\dfrac{\partial^2 r_{12}}{\partial t_1^2}\Delta t_1 + 2\dfrac{\partial^2 r_{12}}{\partial t_1 \partial t_2}\Delta t_2\right)}{1 - \dfrac{1}{c}\dfrac{\partial r_{12}}{\partial t_2} - \dfrac{1}{2c}\dfrac{\partial^2 r_{12}}{\partial t_2^2}\Delta t_2} \tag{13}$$

Let us examine the term

$$\frac{1}{c}\frac{\partial^2 r_{12}}{\partial t_2 \partial t_1}\Delta t_2$$

Now

$$\frac{1}{c}\frac{\partial^2 r_{12}}{\partial t_2 \partial t_1}\Delta t_2 = \frac{\Delta t_2}{c}\left[-\frac{\mathbf{v}_1(t_1)\cdot\mathbf{v}_2(t_2)}{r_{12}} + \frac{\{[\mathbf{r}_2(t_2)-\mathbf{r}_1(t_1)]\cdot\mathbf{v}_1(t_1)\}\{[\mathbf{r}_2(t_2)-\mathbf{r}_1(t_1)]\cdot\mathbf{v}_2(t_2)\}}{r_{12}^3}\right]$$

$\Delta t_2 \approx 1/\nu_T$, and for $\nu_T \approx 890 \cdot 10^6$ cps, $\mathbf{v}_1$ the velocity of the Earth, $\mathbf{v}_2$ the velocity of Venus, we note that

$$\left|\frac{1}{c}\frac{\partial^2 r_{12}}{\partial t_2 \partial t_1}\Delta t_2\right| \approx \frac{2\,(18.5)\,22}{(186{,}000)\,(8.9)\,10^8\,(0.3)\,(93{,}000{,}000)} \approx 10^{-18}$$

so we are justified in omitting this term. A similar argument holds for the other two-second partial derivative terms. Thus

$$\frac{\Delta t_2}{\Delta t_1} = \frac{1 + \dfrac{1}{c}\dfrac{\partial r_{12}}{\partial t_1}}{1 - \dfrac{1}{c}\dfrac{\partial r_{12}}{\partial t_2}} = \frac{1 - \dfrac{[\mathbf{r}_2(t_2)-\mathbf{r}_1(t_1)]\cdot\mathbf{v}_1(t_1)}{c\,r_{12}(t_1,t_2)}}{1 - \dfrac{[\mathbf{r}_2(t_2)-\mathbf{r}_1(t_1)]\cdot\mathbf{v}_2(t_2)}{c\,r_{12}(t_1,t_2)}} \tag{14}$$

with $\mathbf{r}_1(t_1)$ the position vector to P_1 at time t_1, $\mathbf{v}_1(t_1)$, the velocity of P_1 relative to 0 at time t_1, etc., and $r_{12}(t_1, t_2)$ the distance between P_1 at time t_1 and P_2 at time t_2.

The reflected signal will reach P_3 at time t_3 given by

$$t_3 = t_2 + \frac{1}{c} r_{23}(t_2, t_3)$$

A similar argument yields Δt_3 in terms of Δt_2, so that

$$\frac{\Delta t_3}{\Delta t_1} = \frac{\left\{1 - \dfrac{[\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)] \cdot \mathbf{v}_1(t_1)}{c\, r_{12}(t_1, t_2)}\right\} \left\{1 - \dfrac{[\mathbf{r}_3(t_3) - \mathbf{r}_2(t_2)] \cdot \mathbf{v}_2(t_2)}{c\, r_{23}(t_2, t_3)}\right\}}{\left\{1 - \dfrac{[\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)] \cdot \mathbf{v}_2(t_2)}{c\, r_{12}(t_1, t_2)}\right\} \left\{1 - \dfrac{[\mathbf{r}_3(t_3) - \mathbf{r}_2(t_2)] \cdot \mathbf{v}_3(t_3)}{c\, r_{23}(t_2, t_3)}\right\}} \tag{15}$$

Since P_1 is moving relative to 0, its clocks run at a different rate from that of 0, and similarly for P_3. From Eq. (3)

$$\Delta t_1 = \frac{\Delta \bar{t}_1}{\left(1 - \dfrac{v_1^2(t_1)}{c^2}\right)^{1/2}} = \frac{1}{\nu_T} \left(1 - \frac{v_1^2(t_1)}{c^2}\right)^{-1/2}$$

$$\Delta t_3 = \frac{\Delta \bar{t}_3}{\left(1 - \dfrac{v_3^2(t_3)}{c^2}\right)^{1/2}} = \frac{1}{\nu_R} \left(1 - \frac{v_3^2(t_3)}{c^2}\right)^{-1/2}$$

so that

$$\nu_T = \nu_R \left[\frac{1 - \dfrac{v_3^2(t_3)}{c^2}}{1 - \dfrac{v_1^2(t_1)}{c^2}} \right]^{1/2} \frac{\{\;\}\{\;\}}{\{\;\}\{\;\}} \tag{16}$$

the braces being given by Eq. (15).

In order to simplify Eq. (16), we transform all quantities to the single time t_1 and neglect higher order terms. Thus

$$\begin{aligned}
r_{12}(t_1, t_2) &= \{[x_2(t_2) - x_1(t_1)]^2 + \cdots\}^{1/2} \\
&\approx \left\{\left[x_2(t_1) - x_1(t_1) + \frac{dx_2(t_1)}{dt}(t_2 - t_1)\right]^2 + \cdots\right\}^{1/2} \\
&\approx \left\{\left[x_2(t_1) - x_1(t_1) + \frac{r_{12}(t_1)}{c}\,\frac{dx_2(t_1)}{dt}\right]^2 + \cdots\right\}^{1/2} \\
&\approx r_{12}(t_1)\left[1 + 2\left\{\frac{[x_2(t_1) - x_1(t_1)]\,\dfrac{dx_2(t_1)}{dt} + \cdots}{c\,r_{12}(t_1)}\right\}\right]^{1/2} \\
&\approx r_{12}(t_1)\left[1 + \frac{[\mathbf{r}_2(t_1) - \mathbf{r}_1(t_1)]\cdot\mathbf{v}_2(t_1)}{c\,r_{12}(t_1)}\right] \\
&\approx r_{12}(t_1)\left[1 + \frac{1}{c}\dot{R}_2(t_1)\right]
\end{aligned} \tag{17}$$

with $r_{12}(t_1)$ the distance between P_1 and P_2 at time t_1, and $\dot{R}_2(t_1)$ the projection of $\mathbf{v}_2(t_1)$ on the line joining P_1 to P_2 at time t_1, written subsequently as $\dot{R}_{12}(2)$.

Similarly,

$$\mathbf{r}_2(t_2) \approx \mathbf{r}_2(t_1) + \frac{r_{12}(t_1)}{c} \frac{d\mathbf{r}_2(t_1)}{dt} \tag{18}$$

neglecting $(1/c^2)$ terms. Thus

$$1 - \frac{[\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)] \cdot \mathbf{v}_1(t_1)}{c\, r_{12}(t_1, t_2)}$$

$$\approx 1 - \frac{[\mathbf{r}_2(t_1) - \mathbf{r}_1(t_1)] \cdot \mathbf{v}_1(t_1) + \dfrac{r_{12}(t_1)}{c} \mathbf{v}_1(t_1) \cdot \mathbf{v}_2(t_1)}{c\, r_{12}(t_1) \left[1 + \dfrac{1}{c} \dot{R}_2(t_1)\right]}$$

$$\approx 1 - \frac{1}{c} \left\{ \frac{[\mathbf{r}_2(t_1) - \mathbf{r}_1(t_1)] \cdot \mathbf{v}_1(t_1)}{r_{12}(t_1)} + \frac{1}{c} \mathbf{v}_1(t_1) \cdot \mathbf{v}_2(t_1) \right\} \left(1 - \frac{1}{c} \dot{R}_2(t_1)\right)$$

$$\approx 1 - \frac{1}{c} \left\{ \dot{R}_{12}(1) + \frac{1}{c} \mathbf{v}_1(t_1) \cdot \mathbf{v}_2(t_2) \right\} \left(1 - \frac{1}{c} \dot{R}_{12}(2)\right)$$

$$\approx 1 - \frac{1}{c} \dot{R}_{12}(1) + \frac{1}{c^2} \dot{R}_{12}(1)\, \dot{R}_{12}(2) - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} \tag{19}$$

with $\dot{R}_{12}(1)$ the velocity of P_1 projected on the line joining P_1 to P_2 at time t_1, $\dot{R}_{12}(2)$ the velocity of P_2 projected on the line joining P_1 to P_2.

The omission of $(1/c^2)$ terms in Eq. (17) and (18) implies the omission of $(1/c^3)$ terms in Eq. (19).

Similarly

$$1 - \frac{[\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)] \cdot \mathbf{v}_2(t_2)}{c\, r_{12}(t_1, t_2)}$$

$$\approx 1 - \frac{\left[\mathbf{r}_2(t_1) - \mathbf{r}_1(t_1) + \frac{1}{c}\, r_{12}(t_1)\, \mathbf{v}_2(t_1)\right] \cdot \left[\mathbf{v}_2(t_1) + \frac{1}{c}\, r_{12}(t_1)\, \mathbf{a}_2(t_1)\right]}{c\, r_{12}(t_1)\left[1 + \frac{1}{c}\, \dot{R}_2(t_1)\right]}$$

$$\approx 1 - \frac{1}{c}\, \dot{R}_{12}(2) + \frac{1}{c^2}\, \dot{R}_{12}^2(2) - \frac{\mathbf{v}_2^2}{c^2} - \frac{1}{c^2}\, r_{12}(t_1)\, A_2(t_1)$$

with $A_2(t_1)$ the projection of the acceleration vector $\mathbf{a}_2(t_1)$ on the line joining P_1 to P_2, written as $A_{12}(2)$. Thus

$$\left\{1 - \frac{[\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)] \cdot \mathbf{v}_2(t_2)}{c\, r_{12}(t_1, t_2)}\right\}^{-1} \approx 1 + \frac{1}{c}\, \dot{R}_{12}(2) + \frac{\mathbf{v}_2^2}{c^2} + \frac{1}{c^2}\, r_{12}\, A_{12}(2) \tag{20}$$

Further expansions yield

$$1 - \frac{[\mathbf{r}_3(t_3) - \mathbf{r}_2(t_2)] \cdot \mathbf{v}_2(t_2)}{c\, r_{23}(t_2, t_3)} \approx 1 - \frac{1}{c}\, \dot{R}_{23}(2) - \frac{1}{c^2}\, r_{12}\, A_{23}(2) - \frac{1}{c^2}\left(\frac{r_{12} + r_{23}}{r_{23}}\right) \mathbf{v}_2 \cdot \mathbf{v}_3$$

$$+ \frac{1}{c^2}\frac{r_{12}}{r_{23}}\, \mathbf{v}_2^2 + \frac{1}{c^2}\left(\frac{r_{12}+r_{23}}{r_{23}}\right) \dot{R}_{23}(2)\, \dot{R}_{23}(3) - \frac{1}{c^2}\frac{r_{12}}{r_{23}}\, \dot{R}_{23}(2)\, \dot{R}_{23}(2)$$

$$\approx 1 - \frac{1}{c}\, \dot{R}_{23}(2) - \frac{1}{c^2}\, r_{12}\, A_{23}(2) - \frac{2}{c^2}\, \mathbf{v}_2 \cdot \mathbf{v}_3 + \frac{\mathbf{v}_2^2}{c^2} + \frac{2}{c^2}\, \dot{R}_{23}(2)\, \dot{R}_{23}(3)$$

$$- \frac{1}{c^2}\, \dot{R}_{23}(2)\, \dot{R}_{23}(2) \tag{21}$$

since at t_1, $r_{12} = r_{23}$. Also

$$\left\{1 - \frac{[\mathbf{r}_3(t_3) - \mathbf{r}_2(t_2)] \cdot \mathbf{v}_3(t_3)}{c\, r_{23}(t_2, t_3)}\right\}^{-1} \approx 1 + \frac{1}{c} \dot{R}_{23}(3) + \frac{2}{c^2} \mathbf{v}_3^2$$

$$- \frac{1}{c^2} \mathbf{v}_2 \cdot \mathbf{v}_3 + \frac{2}{c^2} A_{23}(3)\, r_{12}$$

$$- \frac{1}{c^2} \dot{R}_{23}^2(3) + \frac{1}{c^2} \dot{R}_{23}(3)\, \dot{R}_{23}(2) \tag{22}$$

It should be noted that, at time t_1, P_1 coincides with P_3, so that $\mathbf{v}_3(t_1) = \mathbf{v}_1(t_1)$, $\dot{R}_{23}(3) = -\dot{R}_{12}(1)$, $\dot{R}_{23}(2) = -\dot{R}_{12}(2)$, $A_{23}(3) = -A_{12}(1)$.

Also

$$\left[\frac{1 - \frac{1}{c^2} \mathbf{v}_3^2(t_3)}{1 - \frac{1}{c^2} \mathbf{v}_1^2(t_1)}\right]^{1/2} \approx \left[\frac{1 - \frac{1}{c^2} \mathbf{v}_1^2(t_1) - \frac{4}{c^3} r_{12}\, \mathbf{v}_1 \cdot \mathbf{a}_1}{1 - \frac{1}{c^2} \mathbf{v}_1^2(t_1)}\right]^{1/2}$$

$$\approx \left[1 - \frac{4}{c^3} r_{12}\, \mathbf{v}_1 \cdot \mathbf{a}_1\right]^{1/2}$$

$$\approx 1 - \frac{2}{c^3} r_{12}\, \mathbf{v}_1 \cdot \mathbf{a}_1$$

$$\approx 1 - 10^{-11} \tag{23}$$

will be replaced by unity.

A final simplification of Eq. (16) yields

$$\nu_T = \nu_R \left\{ 1 + \frac{2}{c} (\dot{R}_{12}(2) - \dot{R}_{12}(1)) + \frac{2}{c^2} [(\mathbf{v}_2 - \mathbf{v}_1)^2 + r_{12}(t_1)(A_{12}(2) - A_{12}(1))] \right\} \tag{24}$$

Let us look at the term $(2/c^2)\, r_{12}\, A_{12}(1)$ when Earth and Venus are in conjunction. Then

$$\frac{2}{c^2} r_{12} A_{12}(1) \approx \frac{2}{(186{,}000)^2} (0.28)(93{,}000{,}000)(3.7)\, 10^{-6} \approx 6 \cdot 10^{-9}$$

Also, $(2/c^2)\, r_{12}\, A_{12}(2) \approx 10^{-8}$, a significant term for the doppler shift.